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On the modeling of collective learning dynamics☆

S. De Lillo [a,b], N. Bellomo [c,*]

[a] Department of Mathematics and Informatics, University of Perugia, Italy

[b] INFN sezione di Perugia, Italy

[c] Department of Mathematics, Politecnico, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

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ABSTRACT

This paper deals with the modeling of the collective learning dynamics of two systems of a heterogeneously distributed population. The first one evolves autonomously towards higher levels of knowledge, while the second system learns from the first one. The approach is based on the mathematical kinetic theory for active particles. The modeling focuses on applications to life sciences.

1. Introduction

This paper deals with the modeling of collective learning processes in large systems of interacting individuals. Specifically, we consider two subsystems such that the first one rapidly evolves in time, while the second one attempts to learn the overall knowledge of the first subsystem by taking advantage of the interactions. The interest in the analysis of this complex dynamics is motivated in various fields of applied and life sciences such as collective learning of populations of robots [1,2], immune competition [3,4], where the immune system learns, by signaling [5] from carriers of a pathology, and transfers the innate defense ability to the acquired one [6]. In some cases, the first population organizes a hiding dynamics to escape the hunting actions of the second population. This is the case of the interaction between detectives and criminals, which has been studied by several authors by means of different mathematical approaches [7–9]. Similarly learning processes can induce a complex dynamics in the social and decisional processes [10,11].

The modeling method proposed in this paper takes advantage of the classical learning theory by Cucker and Smale [12] and of some perspective ideas on the modeling of hiding/learning dynamics announced in [13]. The mathematical background is given by the kinetic theory for active particles [14] originally proposed to model the immune competition [15] and subsequently applied to model a variety of complex systems in social sciences, politics, and opinion formation [16], spread of epidemics [17], and evolution of living systems [18,19]. The collective learning process often has a relevant role in the overall dynamics. More precise definition on the mathematical tools offered by such a theory will be given in the next sections.

The contents of this present paper are developed through three more sections. In detail, Section 2 introduces, at a formal level, the mathematical framework to be properly specialized into models of the aforesaid learning dynamics. Section 3 develops the modeling approach that is based on two sequential steps. The first step concerns the ability of individuals to recognize other individuals of the same or different sub-systems. The second one focuses on the dynamics by which

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* Corresponding author. Tel.: +39 011 564 7514; fax: +39 011 564 7599.
E-mail addresses: silvana.delillo@pg.infn.it (S. De Lillo), nicola.bellomo@polito.it (N. Bellomo).

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individuals increase their own knowledge by taking advantage of the interactions. Section 4 analyzes some technical developments of the mathematical tools looking at applications in life sciences.

This paper aims at developing a new mathematical approach to the modeling of the class of complex systems that have been defined above. The contents are limited to methodological issues, while applications will be treated within a research program, outlined in the last section, initiated by this present paper.

2. Mathematical representation and structures

Let us consider a large system of interacting entities, called *active particles*, grouped into two different subsystems called *functional subsystems*. Active particles have the ability to express a strategy, called *activity*, described by the variable $u \in D_u \subseteq \mathbb{R}_+$, which has a different meaning in each functional subsystem. The activity of individuals is heterogeneously distributed in each functional subsystem, while the overall state is described by the probability distributions:

$$f_i = f_i(t, u) : [0, T] \times D_u \to \mathbb{R}_+, \quad i \in \{1, 2\}, \tag{2.1}$$

where these are divided by the number of particles in each subsystem that are assumed to be constant. Therefore, the f_is are probability densities. It follows that, under suitable, at least locally, integrability assumptions, $f_i(t, u)\,\mathrm{d}u$ denotes, for all $t \geq 0$, the probability of finding an active particle whose state, at time t, is in the elementary volume $[u, u + \mathrm{d}u]$. Moreover for $u^p f_i \in L_1(\mathbb{R}_+)$ calculations of moments, which correspond to the macroscopic description, are obtained as follows:

$$\mathbb{E}_i^p(t) = \int_{\mathbb{R}_+} u^p f_i(t, u)\mathrm{d}u, \tag{2.2}$$

where

$$p = 0 \Rightarrow \mathbb{E}_i^0 = \int_{\mathbb{R}_+} f_i(t, u)\mathrm{d}u = 1, \quad i = 1, 2, \tag{2.3}$$

while higher order moments are computed. The f_is correspond to the state of the *test* particles, which are representative of each subsystem. Moreover, following [14], *candidate* particles, with microscopic state u_*, can acquire, in probability, the state of the *test* particles, after an interaction with *field* particles with microscopic state u^*, while test particles lose their states in their interactions.

The time evolution of f_i is derived by a balance equation of the inlet–outlet flows in the elementary volume $[u, u + \mathrm{d}u]$ of the space of the microscopic states, which can be obtained if the following quantities can be derived by a detailed modeling of interactions at the microscopic level.

- $\eta_{ij} = \eta_{ij}(u^*, u_*|\mathbf{f})$ is the encounter rate between the candidate (or test) active particle, with state u_*, of the i-th functional subsystem and the field active particle, with state u^*, of the j-th functional subsystem.
- $\mathcal{B}_{ij} = \mathcal{B}_{ij}(u_* \to u|u^*, u_*, \mathbf{f})$ is the probability density that a candidate particle, with state u_*, of the i-th functional subsystem ends up into the state u_r of the test particle of the same functional subsystem after the interaction with the field particle, with state u^*, of the j-th functional subsystem.

Both quantities η_{ij} and $\mathcal{B}_{ij}$ can be conditioned by the probability distributions $\mathbf{f} = \{f_1, f_2\}$ of the interacting pairs.

Remark 2.1. $\mathcal{B}_{ij}$ satisfies for all $i, j \in \{1, 2\}$, the following condition:

$$\int_{\mathbb{R}_+} \mathcal{B}_{ij}(u_* \to u|u_*, u^*, \mathbf{f})\mathrm{d}u = 1, \quad \forall u_*,\ u^* \in \mathbb{R}_+, \forall \mathbf{f}. \tag{2.4}$$

The balance of particles in the elementary volume of the the particles yields:

$$\begin{aligned}\partial_t f_i(t, u) &= J_i[\mathbf{f}](t, u) = \sum_{j=1}^{2} J_{ij}[\mathbf{f}](t, u)\\ &= \sum_{j=1}^{2} \int_{\mathbb{R}^+\times\mathbb{R}^+} \eta_{ij}(u_*, u^*|\mathbf{f})\mathcal{B}_{ij}(u_* \to u|u_*, u^*, \mathbf{f}) f_i(t, u_*) f_j(t, u^*)\mathrm{d}u_*\mathrm{d}u^*\\ &\quad - f_i(t, u) \sum_{j=1}^{2} \int_{\mathbb{R}^+} \eta_{ij}(u, u^*|\mathbf{f}) f_j(t, u^*)\mathrm{d}u^*.\end{aligned} \tag{2.5}$$

Eq. (2.5) has been derived under the assumption that the variable u is continuous over $\mathbb{R}^+$. Therefore condition (2.1) implies that the densities f_i decay rapidly to zero at infinity. On the other hand it has been shown [14] that it is useful, in various applications, using a discrete set to identify the activity variable. In fact the state of individuals can be assessed only

by finite intervals rather than precise values. Accordingly, let us consider the set $I_u = \{u_1, \ldots, u_r, \ldots, u_n\}$, so that discrete probability densities follow:

$$f_{ir} = f_i(t, u = u_r) : [0, T] \to \mathbb{R}^+, \quad i \in \{1, 2\},\ r = 1, \ldots, n, \tag{2.6}$$

such that

$$\sum_{r=1}^{n} f_{ir} = 1, \quad \forall\, t \geq 0,\ i \in \{1, 2\},\ r = 1, \ldots, n. \tag{2.7}$$

The interaction terms are defined as follows:

- $\eta_{ij}^{hk} = \eta_{ij}(u_h, u^k|\mathbf{f})$ is the encounter rate between the i-th candidate (or test) active particle, with state u_h and the j-th field active particle, with state u^k.
- $\mathcal{B}_{ij}^{hk}(r) = \mathcal{B}_{ij}(u_h \to u_r|u_h, u^k, \mathbf{f})$ is the probability density that an i-th candidate particle with state u_h ends up into the state u of the test particle of the same functional subsystem after the interaction with the j-th field particle, with state u^k.

The evolution equation is derived by the same balance equation with the following result:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f_{ir}(t) &= Q_i[\mathbf{f}](t) = \sum_{j=1}^{2} Q_{ij}[\mathbf{f}](t) \\ &= \sum_{j=1}^{2} \sum_{h,k=1}^{n} \eta_{ij}(u_h, u^k|\mathbf{f})\, \mathcal{B}_{ij}(u_h \to u_r|u_h, u^k, \mathbf{f})\, f_{ih}(t)\, f_{jk}(t) - f_{ir}(t, u) \sum_{j=1}^{2} \sum_{k=1}^{n} \eta_{ij}(u_i, u^k|\mathbf{f})\, f_{jk}(t),\end{aligned} \tag{2.8}$$

where $\mathbf{f}$ denotes the set of all f_{ih} components of the two probability densities.

3. Modeling methods

The mathematical structure (2.5) can be specialized into a mathematical model when the interaction terms η_{ij} and $\mathcal{B}_{ij}$ are properly modeled according to learning dynamics with respect to a system which evolves in time. The contents of this section presents a methodological approach to model these terms.

Preliminarily, it is necessary to define the distance between the probability densities concerning the two interacting functional subsystems. Let us first consider the continuous case, where condition (2.2) induces the use of the L_1 space. Therefore the distance is the difference of the norm:

$$\mathrm{d}(f_i, f_j)[\mathbf{f}](t) = \|f_i - f_j\|(t) = \int_{\mathbb{R}_+} |f_i(t, u) - f_i(t, u)|\mathrm{d}u, \quad i = 1, 2. \tag{3.1}$$

Analogous calculations in the discrete case yields

$$\mathrm{d}(f_i, f_j)[\mathbf{f}](t) = \|f_i - f_j\|(t) = \sum_{r=1}^{n} |f_{ir}(t) - f_{jr}(t)|, \quad i = 1, 2\ r = 1, \ldots, n. \tag{3.2}$$

3.1. Modeling the encounter rate

The modeling of the encounter rates is based on the concept that these quantities decay with the distance between the interacting active particles. Such a distance depends both on their state and on the distance of their functional subsystems. The following expression summarizes this simple concept in both cases:

$$\eta_{ij} = \eta_{ij}^0 e^{-\alpha_{ij}(1+|u_* - u^*|)(1+\|f_i - f_j\|)}, \tag{3.3}$$

where η_{ij}^0 and α_{ij} are positive constants that characterize the specific system under consideration. These quantities will be defined, respectively, *basic interaction rate* and *decay rate*.

Analogous results are obtained in the discrete state:

$$\eta_{ij}^{hk} = \eta_{ij}^0\ e^{-\alpha_{ij}(1+|u_h - u^k|)(1+\|f_i - f_j\|)}. \tag{3.4}$$

Remark 3.1. This modeling approach corresponds to the assumption that the ability to identify the individual to learn from, is larger within the same subsystem, namely

$$\eta_{ij} = \eta_{ij}^0 e^{-\alpha_{ij}(1+|u_* - u^*|)}, \quad \text{and} \quad \eta_{ij}^{hk} = \eta_{ij}^0 e^{-\alpha_{ij}(1+|u_h - u^k|)}, \tag{3.5}$$

while this ability decreases when the distance increases.

3.2. Modeling the transition probability density

The modeling of the terms $\mathcal{B}_{ij}$ can be obtained by supposing that the output of the interaction is defined by the most probable value m_{ij} and the variance σ_{ij}, which refers to the uncertainty of the output. As a special case $\sigma_{ij} \to 0$, however, due to the conditioning of the probability densities of the interacting active particles the output is still a stochastic game. More precisely, it is assumed that m_{ij} depends on two actions, namely that of the states of the interacting particles and the additional action, at the macroscopic scale, of the particles in the interaction domain $D_u \subseteq \mathbb{R}_+$.

Let us stress, following [20], that candidate or field particles are not sensitive to all particles, but only to those with a state somehow close to them. Moreover, we consider the action applied by the low order moment (for instance the first order moment) of the action of these particles. In particular, the following action, which will be called *effective activity*, is considered:

$$\mathbb{A}[f_j](t) = \frac{\int_{D_u} u f_j(t,u)\mathrm{d}u}{\int_{D_u} f_j(t,u)\,\mathrm{d}u}\,. \tag{3.6}$$

Therefore, the expression of the transition probability density is, for $\sigma_{ij} \to 0$, as follows:

$$\mathcal{B}_{ij} = \delta\left(u - m_{ij}(u_*, u^*, \mathbb{A}[f_j](t))\right), \quad \forall\, u^* \in D_u. \tag{3.7}$$

Calculations in the discrete case are analogous:

$$\mathcal{B}^{hk}_{ij}(r) = \delta\left(u - m_{ij}(u_h, u^k, \mathbb{A}[f_j](t))\right), \qquad \mathbb{A}[f_j](t) = \frac{\sum\limits_{u^k \in I^*} u^k f_{jk}(t)}{\sum\limits_{I*} f_{jk}(t)}, \tag{3.8}$$

for $u^k \in I^*$, where I^* denotes the domain of influence.

Remark 3.2. This modeling approach corresponds to taking into account, similarly to the modeling of the encounter rate, not only the action of the state of the interacting particles, but also that of the *stream* of the states of the field particles acting over the candidate and test particles. Only particles in the influence domain of microscopic states are considered. Specific models can be obtained focusing on well defined systems and learning phenomena. More in general the stream action can include also higher order moments, for instance corresponding to the energy.

3.3. Evolution equations

The modeling approach developed in the preceding section leads to a detailed expression of a class of equations suitable to define the time evolution of the densities f_i is obtained by substituting into (2.5) and (2.8), respectively, the corresponding expression of the encounter rate and of the transition probability density. The formal result is as follows:

$$\begin{aligned}
\partial_t f_i(t,u) &= J_i[\mathbf{f}](t,u) = \sum_{j=1}^{2} J_{ij}[\mathbf{f}](t,u) \\
&= \sum_{j=1}^{2} \eta^0_{ij} \int_{D_u \times D_u} e^{-\alpha_{ij}(1+|u_*-u^*|)(1+\|f_i-f_j\|)}\,\delta\left(u - m_{ij}(u_*, u^*, \mathbb{A}[f_j](t))\right) f_i(t,u_*) f_j(t,u^*)\,\mathrm{d}u_*\,\mathrm{d}u^* \\
&\quad - f_i(t,u) \sum_{j=1}^{2} \eta^0_{ij} \int_{D_u} e^{-\alpha_{ij}(1+|u_*-u^*|)(1+\|f_i-f_j\|)} f_j(t,u^*)\,\mathrm{d}u^*.
\end{aligned} \tag{3.9}$$

Similar calculations can be developed in the case of discrete states with the following result:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f_{ir}(t) &= Q_i[\mathbf{f}](t) = \sum_{j=1}^{2} Q_{ij}[\mathbf{f}](t) \\
&= \sum_{j=1}^{2} \sum_{u^k \in I^*} \eta^0_{ij}\, e^{-\alpha_{ij}(1+|u_h-u^k|)(1+\|f_i-f_j\|)}\,\delta\left(u - m^{hk}_{ij}(u_h, u^k, \mathbb{A}[f_j](t))\right) f_{ih}(t) f_{jk}(t) \\
&\quad - f_{ir}(t,u) \sum_{j=1}^{2} \sum_{I^*} \eta^0_{ij}\, e^{-\alpha_{ij}(1+|u_h-u^k|)(1+\|f_i-f_j\|)} f_{jk}(t).
\end{aligned} \tag{3.10}$$

Some preliminary remarks can now be given in view of the modeling of complex systems that will be developed in the program that has been announced in Section 1. Additional reasonings will be proposed in the next section.

Remark 3.3. The modeling of the learning process can be developed by assuming that the activity variable represents the level of knowledge that is heterogeneously distributed. The learning of the second subsystem from the first one is due both to interaction of a candidate particle with a field particle of the first subsystem with a higher activity, and to interaction with the whole field of the first population represented by the mean value or higher order moments. The output of these interactions consists in an increase, in probability, of the activity variable.

Remark 3.4. If the first population evolves autonomously, it is not sensitive to the interactions with the second one. This means $\eta^0_{12} = 0$, while $\eta^0_{11} > 0$. In general interactions within the same functional subsystem show a higher rate than that of interactions involving different subsystems $\eta^0_{11} > \eta^0_{12}$ and $\eta^0_{22} > \eta^0_{12}$.

Remark 3.5. Interactions are nonlinearly additive, namely the output of the interactions is not the sum of the actions of the field particles over the candidate (or test) particles. In fact only some of the particles are involved (those which are in the interaction domain), while the output of the interaction depends nonlinearly on the state of the interaction particles and on the afore-said streaming effect.

The contents of Remarks 3.3–3.5 can be regarded as preliminary reasonings in view of specific applications. Finally, it is worth mentioning that Eqs. (3.9) and (3.10) have been derived assuming that the number of particles is constant in time. Some applications analyzed in the last section require us to include also the modeling of proliferative and destructive effects.

4. Perspectives

This paper, as already mentioned in Section 1, aims at offering the mathematical basis for the modeling of collective learning processes in large systems of living entities undergoing nonlinearly additive interactions. More precisely, nonlinearity refers both to the encounter rate and to the modification of the state due to interactions.

Such new mathematical tools should allow a revisiting of a variety of models in several fields of life sciences previously derived on the basis of linearly additive interactions. Possibly this refinement may put in evidence emerging behaviors different from those related to the preceding models. Of course, a detailed interpretation of the phenomenology of interactions is needed. The program aims at looking specifically to collective learning phenomena by transferring Remarks 3.3–3.5 into differential models.

In addition to the class of models that have already been mentioned, we wish to mention also swarming phenomena [21–23], where the ability of individuals to learn from the surrounding ones can generate pattern formations that greatly depend on the learning ability.

Focusing on further developments of the mathematical tools, it is worth mentioning that interactions may generate also destructive and/or proliferative events. These, for instance, occur in the immune competition of multicellular systems [3]. In this case the representation of the system is delivered by distribution functions rather than probability densities, while the solution may generate bifurcations [24].

Finally, let us mention that arguably the most challenging perspective consists in introducing a space structure in the arguments of the probability distributions. Developments may look at the deviation of macroscopic equations from the underlying description at the microscopic scale offered by the kinetic theory for active particles in the presence of learning phenomena and nonlinear interactions. This means revisiting the classical literature in the field, among others from [25,26] to [27].

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